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On the Numerical Computation of Eigenvalues and Eigenvectors of Symmetric Integral Equations

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Abstract. The well-known error estimates for the numerical computation of eigenvalues of symmetric integral equations are extended to the computation of the eigenvectors. The results are used to justify the application of an improvement method to obtain an efficient algorithm for solving the eigenvalue problem.

1. Introduction. The eigenvalues and eigenvectors of the integral equation

$$\lambda_n u_n(x) = K u_n(x) \equiv \int_0^1 K(x, t) u_n(t)\, dt, \tag{1}$$

with symmetric kernel $K(x, t) = K(t, x)$, may be approximated numerically by replacing the integral in Eq. (1) by a numerical quadrature. The corresponding equation is

$$\Lambda_n v_n(x) = \sum_{i=1}^{N} w_i K(x, x_i) v_n(x_i), \tag{2}$$

where Λ_n and $v_n(x)$ are approximations to λ_n and $u_n(x)$, respectively. We will make the usual assumptions that $w_i > 0$ and $\sum_{i=1}^N w_i = 1$. Satisfying Eq. (2) at the points x_i, we obtain the matrix eigenvalue problem

$$\Lambda_n \mathbf{v}_n = A \mathbf{v}_n, \tag{3}$$

where $\mathbf{v}_n$ is a vector with components $v_n(x_i)$, and A a matrix with elements

$$a_{ij} = w_j K(x_i, x_j).$$

A, while generally not symmetric, can be symmetrized by a simple similarity transformation, and we will assume that this has been done. For simplicity, we will also assume that the eigenvalues of Eq. (1) are nondegenerate.

Error estimates for the computed eigenvalues have been considered by a number of authors (Wielandt [5], Brakhage [1], Keller [3]). Their results show that the order of accuracy of the computed eigenvalues is the same as the order of accuracy of the chosen quadrature formula. In this paper we show that similar estimates hold for the computed eigenvectors. These estimates are then used as a justification for a method for improving the accuracy of the eigenvalues. In practice this implies that we can obtain highly accurate eigenvalues without having to solve the eigenvalue problem for large matrices.

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2. Error Estimates for the Eigenvalues. Brakhage [1] showed that if Λ_k is an eigenvalue of Eq. (3), then there exists an eigenvalue λ_j of Eq. (1) such that

$$|\Lambda_k - \lambda_j| \leqq \delta/(|\Lambda_k|^2 - \delta)^{1/2}, \tag{4}$$

provided that $|\Lambda_k|^2 > \delta$, where

$$\delta = \max_{0 \leq x, s \leq 1} \left| \int_0^1 K(x, t)K(t, s)\, dt - \sum_{i=1}^N w_i K(x, x_i)K(x_i, s) \right|.$$

The largest eigenvalues are here associated with the smallest error bounds. Intuitively, one expects that the accuracy of the eigenvalue is determined by the smoothness of the associated eigenvector rather than by its magnitude. This is more apparent in Keller's formulation [3] which states that if λ_j is an eigenvalue of Eq. (1) then there exists an eigenvalue Λ_k of Eq. (3) such that

$$|\Lambda_k - \lambda_j| \leqq c \max_{0 \leq x \leq 1} |\bar{e}_j(x)|, \tag{5}$$

where c is a constant (of order unity) and

$$\bar{e}_j(x) = \int_0^1 K(x, t)\bar{u}_j(t)\, dt - \sum_{i=1}^N w_i K(x, x_i)\bar{u}_j(x_i),$$

where $\bar{u}_j$ is an eigenfunction normalized such that

$$\int_0^1 \bar{u}_j^2(x)\, dx = 1.$$

3. Error Estimates for the Eigenvectors. The results of the previous section can be extended to the computation of the eigenvectors. Let $\mathbf{u}_k$ denote the vector with components $u_k(x_i)$. Assume that all the eigenvectors $\mathbf{u}_k$ and $\mathbf{v}_k$ ($k = 1, 2, \cdots, N$) are normalized such that

$$||\mathbf{u}_k|| = \frac{1}{\sqrt{N}} \langle \mathbf{u}_k, \mathbf{u}_k \rangle^{1/2} \equiv \frac{1}{\sqrt{N}} \left\{ \sum_{i=1}^N u_k^2(x_i) \right\}^{1/2} = 1, \qquad ||\mathbf{v}_k|| = 1.$$

Let $\mathbf{e}_j$ denote the vector with components $e_j(x_i)$, where

$$e_j(x) = \int_0^1 K(x, t)u_j(t)\, dt - \sum_{i=1}^N w_i K(x, x_i)u_j(x_i).$$

THEOREM 1. *Let λ_j and u_j be a solution of Eq.* (1), *and let Λ_k be a solution of Eq.* (3) *such that $|\lambda_j - \Lambda_k|$ satisfies the inequality* (5). *Then the eigenvector $\mathbf{v}_k$ associated with Λ_k satisfies*

$$||\mathbf{u}_j - \mathbf{v}_k|| \leqq \frac{1}{\Delta_{jk}} ||\mathbf{e}_j|| + O(||\mathbf{e}_j||^2/\Delta_{jk}^2), \tag{6}$$

where $\Delta_{jk} = \min_{i \neq k} |\Lambda_k - \Lambda_i| - c \max |\bar{e}_j(x)|$, *provided that* $\Delta_{jk} > 0$.

Proof. From Eq. (1) we have

$$\lambda_j \mathbf{u}_j = A\mathbf{u}_j + \mathbf{e}_j. \tag{7}$$

Since A is symmetric its eigenvectors are complete and orthogonal and we can write

$$\mathbf{u}_j = \mathbf{v}_k + \sum_{i=1}^N \alpha_{ji}\mathbf{v}_i, \qquad \mathbf{e}_j = \sum_{i=1}^N \beta_{ji}\mathbf{v}_i.$$

Substituting into Eq. (7),

$$\lambda_j \mathbf{v}_k + \lambda_j \sum_{i=1}^{N} \alpha_{ji}\mathbf{v}_i = \Lambda_k \mathbf{v}_k + \sum_{i=1}^{N} \Lambda_i \alpha_{ji}\mathbf{v}_i + \sum_{i=1}^{N} \beta_{ji}\mathbf{v}_i.$$

Because of the orthogonality of the $\mathbf{v}_i$ we then have

$$\alpha_{jl} = \frac{\beta_{jl}}{\lambda_j - \Lambda_l}, \quad \text{for} \quad l \neq k.$$

Now

$$\begin{aligned} |\lambda_j - \Lambda_l| &\geqq ||\lambda_j - \Lambda_k| - |\Lambda_k - \Lambda_l|| \\ &\geqq \Delta_{jk}, \quad \text{for} \quad \Delta_{jk} > 0. \end{aligned}$$

Also,

$$||e_j||^2 = \sum_{i=1}^{N} (\beta_{ji})^2,$$

so that

$$|\beta_{jl}| \leqq ||\mathbf{e}_j||, \qquad |\alpha_{jl}| \leqq ||\mathbf{e}_j||/\Delta_{jk}, \qquad l \neq k,$$

and since

$$||\mathbf{u}_j||^2 = (1 + \alpha_{jk})^2 + \sum_{i \neq k} (\alpha_{ji})^2 = 1, \qquad \alpha_{jk} = O(||\mathbf{e}_j||^2/\Delta_{jk}^2).$$

Finally,

$$\begin{aligned} ||\mathbf{u}_j - \mathbf{v}_k|| &= \left\{(\alpha_{jk})^2 + \sum_{i \neq k} (\alpha_{ji})^2\right\}^{1/2} \\ &\leqq \left\{(\alpha_{jk})^2 + \frac{1}{\Delta_{jk}^2} \sum_{i \neq k} (\beta_{ji})^2\right\}^{1/2} \leqq \frac{1}{\Delta_{jk}} ||\mathbf{e}_j|| + O(||\mathbf{e}_j||^2/\Delta_{jk}^2). \end{aligned}$$

Thus, if Λ_k is well separated from the other computed eigenvalues the norm of the error is essentially proportional to the quadrature error. From (6) we have immediately that

$$|u_j(x_i) - v_k(x_i)| \leqq \frac{\sqrt{N}}{\Delta_{jk}} ||\mathbf{e}_j|| + O(||\mathbf{e}_j||^2/\Delta_{jk}^2) \tag{8}$$

at the meshpoints x_i, $i = 1, 2, \cdots, N$. If $v_k(x)$ is computed by Eq. (2), then a similar estimate holds for all points of the range.

THEOREM 2. *If $K(x, t)$ is bounded in the closed unit square, $|K(x, t)| \leqq L$, then, for all $x \in [0, 1]$,*

$$\begin{aligned} |u_j(x) - v_k(x)| \leqq{} & \frac{1}{|\lambda_j|} \max |e_j(x)| + \frac{c}{|\Lambda_k|} \max |\bar{e}_j(x)| \, |u_j(x)| \\ & + \frac{L\sqrt{N}}{|\Lambda_k| \, \Delta_{jk}} ||\mathbf{e}_j|| + O(||\mathbf{e}_j||^2/\Delta_{jk}^2), \end{aligned} \tag{9}$$

where c is the same constant as in Eq. (5).

Proof.

$$\begin{aligned}|u_j(x) - v_k(x)| &\leqq \left|\frac{1}{\lambda_j}\int_0^1 K(x, t)u_j(t)\,dt - \frac{1}{\Lambda_k}\sum_{i=1}^N w_i K(x, x_i)v_k(x_i)\right| \\ &\leqq \left|\frac{1}{\lambda_j}\int_0^1 K(x, t)u_j(t)\,dt - \frac{1}{\lambda_j}\sum_{i=1}^N w_i K(x, x_i)u_j(x_i)\right| \\ &\quad + \left|\frac{1}{\lambda_j}\sum_{i=1}^N w_i K(x, x_i)u_j(x_i) - \frac{1}{\Lambda_k}\sum_{i=1}^N w_i K(x, x_i)u_j(x_i)\right| \\ &\quad + \left|\frac{1}{\Lambda_k}\sum_{i=1}^N w_i K(x, x_i)u_j(x_i) - \frac{1}{\Lambda_k}\sum_{i=1}^N w_i K(x, x_i)v_k(x_i)\right|\end{aligned}$$

and, applying inequalities (5) and (8), the estimate (9) follows.

4. An Efficient Method for Computing the Eigenvalues. We denote the inner product

$$\int_0^1 f(t)g(t)\,dt$$

by (f, g). It is well known that if $v_k(x)$ is an approximation to $u_k(x)$ such that $v_k(x) = u_k(x) + O(\Delta)$, then an approximate eigenvalue may be computed by the Rayleigh quotient

$$\hat{\Lambda}_k = (v_k, Kv_k)/(v_k, v_k) = \lambda_k + O(\Delta^2). \tag{10}$$

Since a solution of Eq. (3) produces eigenvalues and eigenvectors with the same order of accuracy we can use the computed eigenvectors to improve the accuracy of the eigenvalues. Normally one will not be able to evaluate the required integrals in closed form, but Eq. (10) may be replaced by its discrete analogue

$$\bar{\Lambda}_k = \frac{\tilde{\mathbf{v}}_k^T A_M \tilde{\mathbf{v}}_k}{\tilde{\mathbf{v}}_k^T \tilde{\mathbf{v}}_k}, \tag{11}$$

where A_M is the $M \times M$ matrix with elements

$$a_{ij} = \bar{w}_j K(\bar{x}_i, \bar{x}_j),$$

and $\tilde{\mathbf{v}}_k$ is the vector with components $v_k(\bar{x}_i)$, $\bar{x}_i$ being the quadrature points for the M-point quadrature formula, and $\bar{w}_i$ the associated weights. The $v_k(\bar{x}_i)$ are to be obtained from $v_k(x_i)$ by means of Eq. (2). It can then be shown that

$$\bar{\Lambda}_k = \Lambda_k^M + O(\|\mathrm{e}_j\|^2),$$

where Λ_k^M is the eigenvalue we would obtain by solving Eq. (3) with M quadrature points. If we want to find eigenvalues to a certain accuracy we can use the following procedure:

(1) Choose a quadrature formula with N points and solve (3) to find the eigenvalues and eigenvectors. The N in this case can be kept considerably smaller than what we would have to choose if the results of (3) were to be used directly.

(2) Use Eq. (11) with $M > N$ to evaluate improved eigenvalues. This step involves matrix multiplications only. Provided our initial approximation to the eigenvalues is

TABLE 1

	Example 1	Example 2	Example 3	Example 4
$N = 2$	.42678	1.33222	.25000	.27192
$N = 10$	.40612	1.35214	.34343	.24407
$N = 20$	.40549	1.35281	.34641	.24324
$N = 30$	.40538	1.35293	.34697	.24309
$N = 50$	.40532	1.35299	.34725	.24301
$N = 2, \quad M = 50$	.40507	1.35293	.34482	.24261
$N = 10, \quad M = 50$	.40532	1.35299	.34725	.24300
exact λ_0	.40528	1.35303	.34741	.24296

sufficiently accurate, the final eigenvalues will be essentially as accurate as if we had solved (3) using M points.

Thus, we avoid what is frequently the most time consuming part of the computation, the solution of the eigenvalue problem (3) for large matrices.

5. Numerical Examples. In the following examples we computed the largest eigenvalue λ_0 by the midpoint quadrature rule using values of $N = 2, 10, 20, 30, 50$ with Eq. (3), and values of $N = 2, 10$, $M = 50$ with the indicated improvement method.

Example 1.

$$K(x, t) = \min (x, t), \qquad \lambda_0 = 4/\pi^2 = .40528.$$

Example 2 (*Brakhage* [1]).

$$K(x, t) = e^{xt}, \qquad \lambda_0 = 1.35303.$$

Example 3 (*Bückner* [2, *p*. 49]).

$$K(x, t) = |x - t|, \qquad \lambda_0 = .34741.$$

Example 4 (*Mikhlin* [4, *p*. 86]).

$$\begin{aligned} K(x, t) &= \tfrac{1}{2}x(2 - t), \qquad x \leqq t, \\ &= \tfrac{1}{2}t(2 - x), \qquad x \geqq t, \end{aligned}$$

$$\lambda_0 = .24296.$$

Results are given in Table 1.

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